EX Find $x(t)$ at time t for the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

& $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $u(\tau) = 0$

sol

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}, \ |\lambda I - A| = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2$$

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \ P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\Lambda = P^{-1} A P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$e^{At} = P e^{\Lambda t} P^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} e^{t} + \frac{1}{3} e^{2t} & -\frac{2}{3} e^{t} + \frac{2}{3} e^{2t} \\ -\frac{1}{3} e^{t} + \frac{1}{3} e^{2t} & -\frac{1}{3} e^{t} + \frac{2}{3} e^{2t} \end{bmatrix}$$

$$x_h(t) = e^{At} * x(0)$$

$$= \begin{bmatrix} \frac{2}{3} e^{t} + \frac{1}{3} e^{2t} & -\frac{2}{3} e^{t} + \frac{2}{3} e^{2t} \\ -\frac{1}{3} e^{t} + \frac{1}{3} e^{2t} & -\frac{1}{3} e^{t} + \frac{2}{3} e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_h(t) = \begin{bmatrix} e^{2t} \\ -\frac{2}{3} e^{t} + e^{2t} \end{bmatrix}$$

$$x_p(t) = \int_0^t e^{A(t-\tau)} * B * u(\tau)\, d\tau$$

$$= \int_0^t \begin{bmatrix} \frac{2}{3} e^{-(t-\tau)} + \frac{1}{3} e^{-2(t-\tau)} & -\frac{2}{3} e^{-(t-\tau)} + \frac{2}{3} e^{-2(t-2)} \\ -\frac{1}{3} e^{-(t-2)} + \frac{1}{3} e^{-(t-2)} & -\frac{1}{3} e^{-(t-2)} + \frac{2}{3} e^{-2(t-\tau)} \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} * 0$$

$$= [0]$$

EX find $x(t)$ for the System below?.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

at $t = 0 \Rightarrow u(\tau) = 1 \,,\; x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

sol $|\lambda I - A| = 0 \Rightarrow \lambda_1 = -2 ,\; \lambda_2 = -1$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\Lambda = P^{-1} A P = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e^{At} = P e^{\Lambda t} P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} * \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

$$x_h(t) = e^{At} \cdot x(0) = [0]$$

$$x_p(t) = \int_0^t \begin{bmatrix} -e^{-2(t-\tau)} + 2e^{-(t-\tau)} & -e^{-2(t-\tau)} + e^{-(t-\tau)} \\ 2e^{-2(t-\tau)} - 2e^{-(t-\tau)} & 2e^{-2(t-\tau)} - e^{-(t-\tau)} \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} -e^{-2(t-\tau)} + e^{-(t-\tau)} \\ 2e^{-2(t-\tau)} - e^{-(t-\tau)} \end{bmatrix} d\tau$$

$$x_p = \begin{bmatrix} \dfrac{-e^{-2(t-\tau)}}{2} \Big|_0^t + e^{-(t-\tau)} \Big|_0^t \\ e^{-2(t-\tau)} \Big|_0^t - e^{-(t-\tau)} \Big|_0^t \end{bmatrix}$$

$$\therefore x_p = \begin{bmatrix} -\frac{1}{2}\left[e^{-2\times 0} - e^{-2t}\right] + \left[e^{0} - e^{-t}\right] \\ \left[e^{0} - e^{-2t}\right] - \left[e^{0} - e^{-t}\right] \end{bmatrix}$$

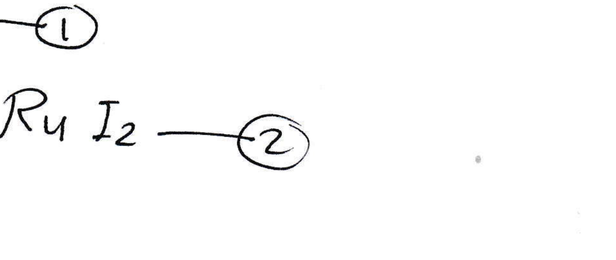
$$= \begin{bmatrix} -\frac{1}{2}(1-e^{-2t}) + 1 - e^{-t} \\ 1 - e^{-2t} - 1 + e^{-t} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}e^{-2t} + 1 + e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Application of matrices to electric Circuit :-

EX Find I_1 & I_2 for this circuit



sol

$$5 = R_1 I_1 + R_2 I_1 - R_2 I_2 \quad \text{—— (1)}$$

$$0 = R_2 I_2 + R_2 I_1 + R_3 I_2 + R_4 I_2 \quad \text{—— (2)}$$

$$\therefore \quad 5 = 3 I_1 - 2 I_2 \quad \text{—— (1)}$$

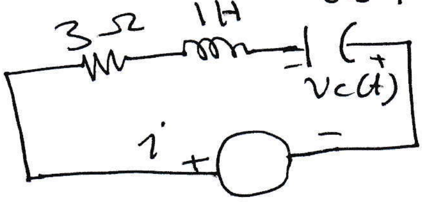
$$0 = -2 I_1 + 6 I_2 \quad \text{—— (2)}$$

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}, \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

By Gramer rule :-

$$I_1 = \frac{\begin{vmatrix} 5 & -2 \\ 0 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix}} = \frac{30}{14} = \frac{15}{7} , \quad I_2 = \frac{\begin{vmatrix} 3 & 5 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix}} = \frac{5}{7}$$

EX For the network shown, write the state equation & then find the current $i(t)$ & the voltage $v_c(t)$, if $i(0) = 0$, $v_c(0) = 1$ & $v(t) = u(t)$, $u(0) = 0$



sol

$$Ri + L\frac{di}{dt} + v_c = v(t) \quad \text{—— (1)}$$

$$i(t) = X_1 , \quad v_c(t) = X_2 \quad \text{—— (2)}$$

$$3X_1 + \dot{X}_1 + X_2 = v(t) , \quad 0.5\dot{X}_2 = X_1$$

$$\dot{X}_1 = -3X_1 - X_2 + u(t) , \quad \dot{X}_2 = 2X_1$$

$v_c(t) = \int \frac{i}{c}\, dt$

$dv_c(t) = \frac{i}{c}$

$i = c * \frac{dv_c(t)}{dt}$

$\therefore X_1 = 0.5X_2$

$$[\dot{X}] = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$|\lambda I - A| = 0 \Rightarrow \lambda_1 = -1 , \quad \lambda_2 = -2$$

$$P = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} , \quad P^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\Lambda = P^{-1}AP = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\therefore \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad \therefore e^{At} = P e^{\Lambda t} P^{-1}$$

$$e^{At} = \begin{bmatrix} -e^{-t}+2e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}+2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}$$

$$x_h(t) = e^{At} \cdot X(0) = \begin{bmatrix} -e^{-t}+2e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}+2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad x(t) = \begin{bmatrix} -e^{-t}+e^{-2t} \\ 2e^{-t}-e^{-2t} \end{bmatrix}, \quad x_p = 0$$

Symmetric, Skew-Symmetric, and Orthogonal matrices.

A real square matrix $A = [a_{jk}]$ is called,

1- "Symmetric" if transposition leaves it unchanged,

$$A^T = A \quad \text{thus} \quad a_{kj} = a_{jk},$$

2- "Skew-Symmetric" if transposition gives the negative of A,

$$A^T = -A \quad \text{thus} \quad a_{kj} = -a_{jk},$$

3- "orthogonal" if transposition gives the inverse of A,

$$A^T = A^{-1}.$$

Ex Symmetric matrix

$$A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

$$\therefore A^T = A$$

Ex Skew-Symmetric matric

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

$$-A = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix}$$

$$\therefore \quad A^T = -A$$

Ex Orthogonal Matrix

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

$$\therefore \quad A^T = A^{-1}$$

Any real square matrix $\tilde{A}$ may be written as the sum of a symmetric matrix "$\tilde{R}$" and a skew-symmetric matrix "$\tilde{S}$", where.

$$\tilde{R} = \frac{1}{2}(A + A^T) \text{ and } \tilde{S} = \frac{1}{2}(A - A^T).$$

$$A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix}$$

$$(A + A^T) = \begin{bmatrix} 18 & 7 & 7 \\ 7 & 6 & -4 \\ 7 & -4 & 6 \end{bmatrix}$$

$$R = \frac{1}{2}(A + A^T)$$

$$R = \begin{bmatrix} 9 & 7/2 & 7/2 \\ 7/2 & 3 & -2 \\ 7/2 & -2 & 3 \end{bmatrix}$$

$$(A - A^T) = \begin{bmatrix} 0 & 3 & -3 \\ -3 & 0 & -12 \\ 3 & 12 & 0 \end{bmatrix}$$

$$S = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 3/2 & -3/2 \\ -3/2 & 0 & -6 \\ 3/2 & 6 & 0 \end{bmatrix}$$

$$A = R + S \Rightarrow A = \begin{bmatrix} 9 & 7/2 & 7/2 \\ 7/2 & 3 & -2 \\ 7/2 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 3/2 & -3/2 \\ -3/2 & 0 & -6 \\ 3/2 & 6 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$$

* Eigenvalues of Symmetric and Skew-Symmetric matrices.

a) The eigenvalues of a Symmetric matrix are real.

b) the eigenvalues of a skew-Symmetric matrix are pure imaginary or Zero.

Ex

Symmetric matric $A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$

$\lambda_1 = -6.12$, $\lambda_2 = 0.28$, $\lambda_3 = 6.86$

skew-Symmetric matrix $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

$\lambda_1 = 0$, $\lambda_2 = 25i$, $\lambda_3 = -25i$

Orthogonal Transformations and Orthogonal Matrices

Orthogonal transformations are transformations

$$y = Ax$$ where A is an orthogonal matrix.

With each vector x in R^n Such a transformation assigns a vector y in R^n. For instance, the plane rotation through an angle θ

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

* if A is onthogonal matrix, then

$$A^{-1}A = A^{T}A = I$$

* the determinant of an onthogonal matrix has the value $+1$ or -1.

* the eigenvalues of an onthogonal matrix A are real or Complex Conjugates in pairs and have absolute value 1.

ex Onthogonal matrix $A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$.

$$\lambda_1 = -1 \,,\ \lambda_2 = \frac{5}{6} + \frac{\sqrt{11}}{6} i \,,\ \lambda_3 = \frac{5}{6} - \frac{\sqrt{11}}{6} i$$

$$A^{-1}A = A^{T}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Proof

$AX = \lambda X$ where λ is eigen value

Proof $P^{-1}AX = \lambda P^{-1}X$,

we have $P^{-1}AX \Rightarrow I = PP^{-1}$

$P^{-1}AIX \Rightarrow P^{-1}APP^{-1}X \Rightarrow (P^{-1}AP)P^{-1}X$

$\Rightarrow \lambda P^{-1}X$ that what need.

<u>Ex</u> $A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

$\lambda = P^{-1} A P \Rightarrow \lambda = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$y_1 = P^{-1} x_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$y_2 = P^{-1} X_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Indeed, thes are eigenvectors of the diagonal matrix λ

* Proof if $D = X^{-1} A X$, then $D^2 = X^{-1} A^2 X$

$D^2 = DD = (X^{-1} A X)(X^{-1} A X) = X^{-1} A (X X^{-1}) A X$

$= X^{-1} A A X$

$= X^{-1} A^2 X$

Quadratic Forms :-

By definition, a quadratic form Q in the components $x_1, \dots, x_n$ of a vector x is a sum of "n^2" terms, namely,

$$Q = x^T A X = \sum_{j=1}^{n} \sum_{K=1}^{n} a_{jK} x_j x_K \quad (a_{Kj} = a_{jK})$$

$$= a_{11} x_1^2 + a_{12} x_1 x_2 + \cdots + a_{1n} x_1 x_n + a_{21} x_2 x_1 +$$
$$a_{22} x_2^2 + \cdots + a_{2n} x_2 x_n + \cdots a_{n1} x_n x_1 +$$
$$a_{n2} x_n x_2 + \cdots + a_{nn} x_n^2 .$$

$A = [a_{jK}]$ is called the "Coefficient matrix" of the form. We may assume that A is "Symmetric".

EX $A = \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}$, Quadratic Form. Symmetric coefficient matrix

$$x^T A X = [x_1 \;\; x_2] \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3x_1^2 + 4x_1 x_2 + 6 x_1 x_2 + 2x_2^2$$

$$= 3x_1^2 + 10 x_1 x_2 + 2x_2^2$$, Here $4 + 10 = 10 = 5 + 5$

From the Corresponding "Symmetric" matrix $C = [c_{jK}]$, where $C_{jK} = \frac{1}{2} [a_{jK} + a_{Kj}]$, thus $C_{11} = 3$, $C_{12} = C_{21} = 5$, $C_{22} = 2$, we get the same result, indeed,

$$x^T C x = [x_1 \ x_2]\begin{bmatrix}3 & 5\\ 5 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$
$$= 3x_1^2 + 10x_1x_2 + 2x_2^2.$$

Quadratic forms occur in physics and geometry, for instance, in connection with conic sections (ellipse $x_1^2/a^2 + \frac{x_2^2}{b} = 1$, etc). their transformation to principal axes is an important practical task related to the diagonalization of matrices, as follows.

- For the symmetric coefficient matrix "A"

$$X^{-1} = X^T$$

we have $D = X^{-1}AX$

or $A = XDX^{-1} = XDX^T$, sub into $\boxed{Q = x^T A x}$

$$Q = x^T X D X^{-1} x.$$

if we set $X^T x = y$, then, since $X^T = X^{-1}$, we have

$X^{-1}x = y$, and thus obtain

$$x = Xy$$

we have $x^T X = (X^T x)^T = y^T$ and $X^T x = y$, so that Q becomes simply

$$Q = y^T D y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

Canonical form.

* Principal Axes theorem

$$Q = x^T A x = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k \quad (a_{kj} = a_{jk})$$

Where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the (symmetric) matrix A, and X is an orthogonal matrix with corresponding eigenvectors $x_1, \dots, x_n$, respectively, as Column vectors

Ex Find out what type of conic section the following quadratic form represents and transform it to principal axes:

$$Q = 17x_1^2 - 30x_1 x_2 + 17x_2^2 = 128.$$

Sol

We have $Q = x^T A x$, where

$$A = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

characteristic equation $(17-\lambda)^2 - 15^2 = 0$.

$$\lambda_1 = 2, \lambda_2 = 32.$$

for $Q = y^T D y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$.

$$Q = 2y_1^2 + 32y_2^2$$

we have $Q=128$ represents the ellipse $2y_1^2+32y_2^2=128$,

that is $\frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$.

for determining the direction of the principal axes in the $x_1 x_2$ - coordinates, we have to determine normalized eigenvectors from $(\cancel{\lambda} \lambda I - A)x=0$ with $\lambda_1=2$, and $\lambda_2=32$ and

then use $\quad x = Xy$

$$X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad X_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$x = Xy = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{2}} - \frac{y_2}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{2}}$$

this is a 45° rotation, these results agree with

* if $A = \begin{bmatrix} 3+4i & 1-i \\ 6 & 2-5i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 3-4i & 1+i \\ 6 & 2+5i \end{bmatrix}$ and

$$\bar{A}^T = \begin{bmatrix} 3-4i & 6 \\ 1+i & 2+5i \end{bmatrix}$$

Hermitian, Skew-Hermitian, and Unitary Matrices

A square matrix $A = [a_{kj}]$ is called

Hermitian if $\bar{A}^T = A$, that is $\bar{a}_{kj} = a_{jk}$

Skew-Hermitian if $\bar{A}^T = -A$, that is, $\bar{a}_{kj} = -a_{jk}$

Unitary if $\bar{A}^T = A^{-1}$.

EX

$$A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$$

$\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix} \Rightarrow \bar{A}^T = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ ← Hermitatan.

$-A = \begin{bmatrix} -4 & -1+3i \\ -1-3i & -7 \end{bmatrix}$, $\bar{A}^T \neq -A$ not Skew-Hermitian.

$A^{-1} = \begin{bmatrix} -0.38 & -0.85+0.16i \\ -0.05-0.16i & 0.222 \end{bmatrix}$, $\bar{A}^T \neq A^{-1}$ not Unitary.

$$B = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix} \Rightarrow \bar{B}^T = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix}$$

$\bar{B}^T \neq B$ not Hermitian

$$-B = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix}$$

$\therefore$ $\bar{B}^T = -B$ Skew-Hermitian

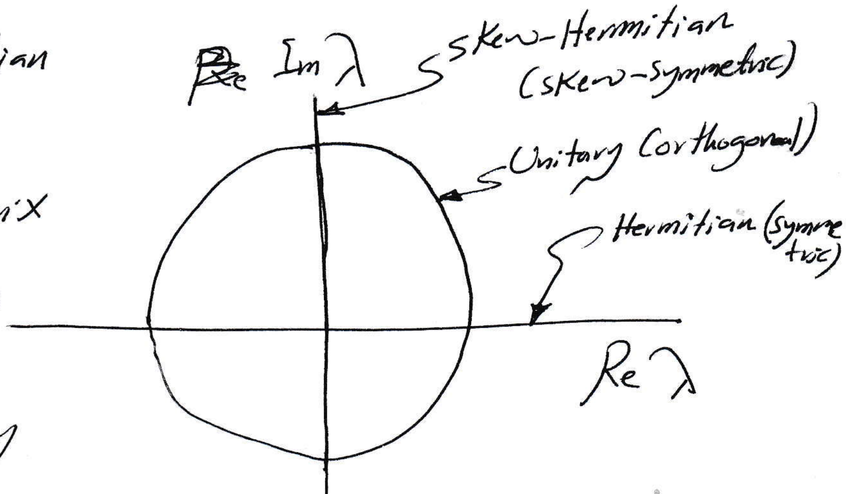
you must do the rest of matrices.

Description of Eigenvalues of matrices

a) the eigenvalues of a Harmitian matrix are real.

b) the " " " skew matrix are pure imajinary or zero

c) " " " a unitary matrix have absolute value 1.



Location of the eigenvalues of Hermitian, Skew-Hermitian, and Unitary matrices in the complex λ-plane.

- for matrix A, $\lambda_1 = 9$, $\lambda_2 = 2$
- for matrix B, $\lambda_1 = 4i$, $\lambda_2 = -2i$
- for matrix C, $\lambda_1 = 0.866 + 0.5i$
  $\lambda_2 = -0.866 + 0.5i$

$|\pm 0.866 + 0.5i|^2 = 1$

Matrices

Homework 1

1- if $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 2 & -1 \end{bmatrix}$, find:-

1) $A+B$ 2) $3B$ 3) $-2B$ 4) $A+2B$ 5) $2A+B$

6) $A-B$ 7) $A-2B$ 8) $B-A$ 9) A^T 10) B^T

11) Show that $(A+B)^T = A^T + B^T$ 12) $C = A^T B$

13) C^{-1} 14) Show that $CC^{-1} = C^{-1}C = I$.

Matrices   Homework 2

① Solve the following set of three equations by Gramer and by A^{-1}.

$$x_1 - x_2 - 2x_3 = 1$$
$$2x_1 \quad - x_3 = 2$$
$$x_1 + x_2 + x_3 = 3$$

② Solve the following set of two equations by Gramer's rule and by A^{-1}

$$2x_1 + 3x_2 = 8$$
$$x_1 - x_2 = 1$$

③ find the eigen value and eigen vector and A for the following matrices:-

ⓐ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, ⓑ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$, ⓒ $A = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}$

ⓓ $A = \begin{bmatrix} -15 & -14 & -40 \\ 6 & 7 & 14 \\ 5 & 4 & 14 \end{bmatrix}$, ⓔ $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, ⓕ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 7 & 0 \end{bmatrix}$

ⓖ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & 16 & -1 \end{bmatrix}$, ⓗ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -125 & -75 & -15 \end{bmatrix}$

④ Find the variable of matrix B according to equation $D^2 - 4D + 4I^2 = B$, where $D = P^{-1}AP$ and P is the eigen vector of matrix A

$$A = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} K_1 & K_2 \\ K_3 & 7K_4 \end{bmatrix}.$$

⑤ Find A^{25}, A^{50}, e^{A}, e^{At} for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

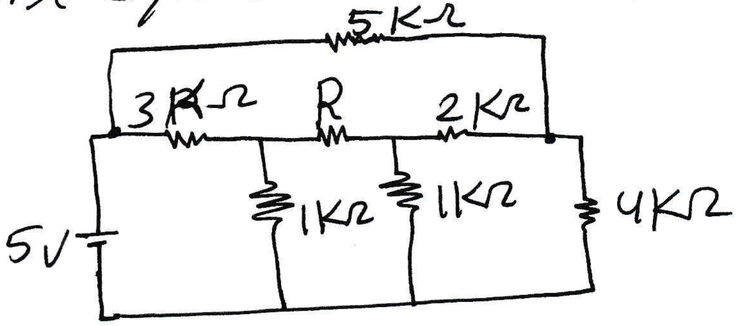
⑥ Find A^{5} and A^{-1} by Hamilton theorem for

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

Matrices

Homework 3

Q1/ Find the value of R where the all determinant matrix equal 669 if $I_4 = 0.7mA$



Q2/ Find I_1, I_2, and I_3 for Figure below.

$R_1 = 1K\Omega$

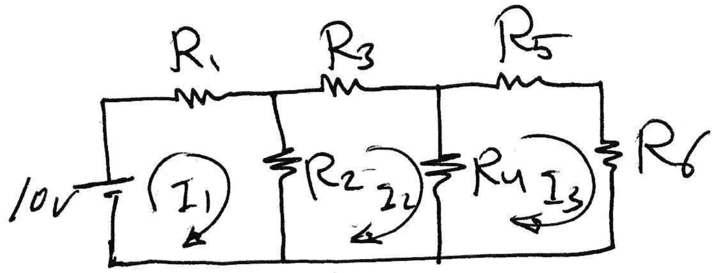
$R_2 = 2K\Omega$

$R_3 = 3K\Omega$

$R_4 = 2K\Omega$

$R_5 = 5K\Omega$

$R_6 = 6K\Omega$



Matrices

Homework 4

1- Are the following matrices Symmetric, skew-Symmetric, or orthogonal?

(a) $A = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$, (b) $A = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$, (c) $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$, (e) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

(f) $A = \begin{bmatrix} 4/9 & 8/9 & 1/9 \\ -7/9 & 4/9 & -4/9 \\ -4/9 & 1/9 & 8/9 \end{bmatrix}$.

Matrices

Homework 5

1- Verify this for A and $A = P^{-1}AP$. If y is an eigenvector of P, show that $x = Py$ are eigenvectors of A. Show the details of your work.

a) $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$, $P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix}$

c) $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{bmatrix}$

d) $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

e) $A = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2- Find an eigenbasis (a basis of eigenvectors) and diagonalize. Show the details.

a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, b) $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, c) $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

d) $A = \begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix}$, e) $A = \begin{bmatrix} -5 & -6 & 6 \\ -9 & -8 & 12 \\ -12 & -12 & 16 \end{bmatrix}$, $\lambda_1 = -2$

f) $A = \begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{bmatrix}$, $\lambda_1 = 10$

g) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

3) What kind of conic section is given by the quadratic form? Transform it to principal axes. Express $x^T = [x_1 \; x_2]$ in terms of the new coordinate vector $y^T = [y_1 \; y_2]$, as in example we took before.

a) $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

b) $3x_1^2 + 8x_1x_2 - 3x_2^2 = 10$

c) $3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$

d) $9x_1^2 + 6x_1x_2 + x_2^2 = 10$

e) $x_1^2 - 12x_1x_2 + x_2^2 = 70$

f) $4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$

g) $-11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$